Chapter 22 – Gauss' Law and Flux

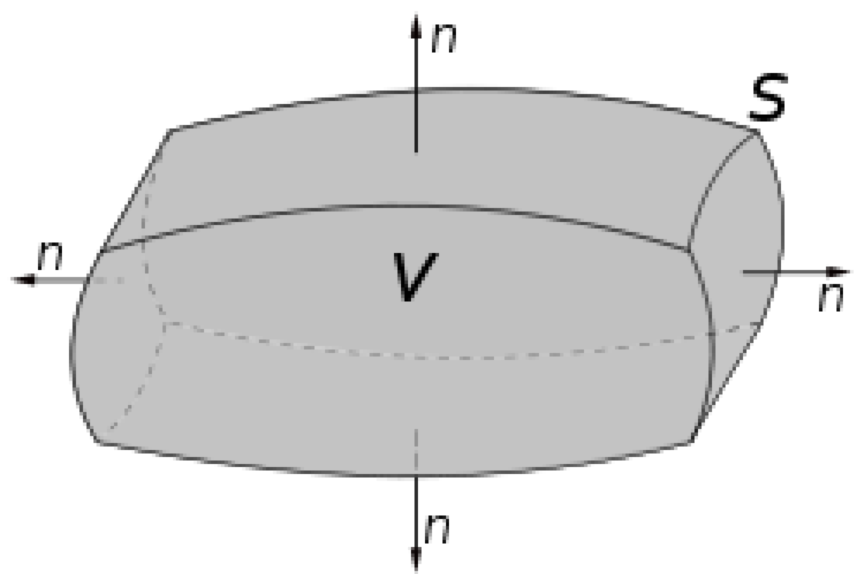
- Lets start by reviewing some vector calculus
- Recall the divergence theorem
- It relates the "flux" of a vector function **F** thru a closed simply connected surface S bounding a region (interior volume) V to the volume integral of the **divergence** of the function **F**
- **Divergence F => ∇•F**

$$\iiint\limits_V (\nabla \cdot \mathbf{F})\, dV = \oiint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

Volume integral of divergence of **F** = Surface (flux) integral of **F**

Mathematics vs Physics

- There is NO Physics in the previous "divergence theorem" known as Gauss' Law
- It is purely mathematical and applies to ANY well behaved vector field **F**(x,y,z)



Some History – Important to know

- First "discovered" by Joseph Louis Lagrange 1762
- Then independently by Carl Friedrich Gauss 1813
- Then by George Green 1825
- Then by Mikhail Vasilievich Ostrogradsky 1831
- It is known as Gauss' Theorem, Green's Theorem and Ostrogradsky's Theorem
- In Physics it is known as Gauss' "Law" in Electrostatics and in Gravity (both are inverse square "laws")
- It is also related to conservation of mass flow in fluids, hydrodynamics and aerodynamics
- Can be written in integral or differential forms

Integral vs Differential Forms

- Integral Form $$\iiint_V (\nabla \cdot \mathbf{F})\, dV = \oiint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$
- Differential Form (we have to add some Physics)
- Example - If we want mass to be conserved in fluid flow – ie mass is neither created nor destroyed but can be removed or added or compressed or decompressed then we get
- **Conservation Laws**

Continuity Equations – Conservation Laws

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Conservation of mass in compressible fluid flow
ρ = fluid density, **u** = velocity vector

$$\nabla \cdot \mathbf{u} = 0$$

Conservation of an incompressible fluid
ρ= fluid density = constant here

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0,$$

Conservation of charge in electric current flow
J = current flux vector (amps/m^2) , ρ = charge density (coulombs/ m^3)

$$\nabla \cdot \mathbf{j} = -\frac{\partial}{\partial t} P(x,t)$$

Conservation of probability on Quantum Mechanics, **j** = probability flux vector,
ρ = probability

$$\frac{\partial \varphi}{\partial t} + \nabla \cdot f = s$$

General Continuity Equation with source term s = source or sink – creation or annihilation

$$\frac{\partial \varphi}{\partial t} + \nabla \cdot f = 0.$$

General Continuity Equation with s = 0

What are Continuity, Conservation Laws?

The equation

$$\iiint_V (\nabla \cdot \mathbf{F})\, dV = \oiint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

What it means

- Let **F** = ρ**V** for a fluid then
- **F** = flux (mass flow) of fluid per unit area per unit of time (Kg/s-m^2)
- If you integrate this over a closed surface (right hand side) you get the net mass change per unit time going INTO or OUT OF the surface
- This must be -$\partial m/\partial t$ where m = mass inside the surface. Note minus sign – this depends on how we define the outward normal
- BUT m = $\int \rho\, dV$ and $\partial m/dt = \int \partial\rho/\partial t\, dV$
- Now equate the two sides of the equation
- We now get $\nabla \bullet \mathbf{F} = -\partial\rho/\partial t$ or
- $\nabla \bullet \mathbf{F} + \partial\rho/\partial t = 0$ => Continuity equation

Gauss' Law in Electromagnetism

- We start with an assumption about the **E** field from a point source.
- Assume it obeys Coulomb's Law – ie inverse square law

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{e_r}}{r^2}$$

Where $\mathbf{e}_r$ is a radial unit vector away from the point charge q

Compute the surface integral of **E(r)** over a sphere of radius r with the charge q at the **center.** We will then use Gauss' Law.

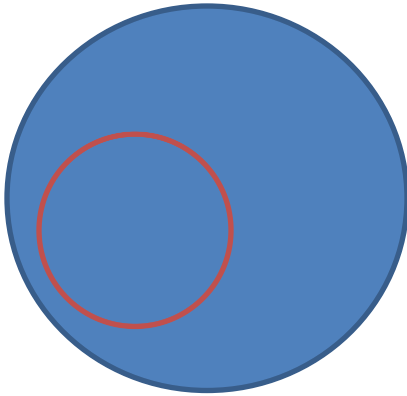
Surface integral over sphere

- Compute the surface integral of **E(r)** over a sphere of radius r with the charge q at the **center.**
- $\int E(r) \bullet dA = 4\pi r^2 \; * \; kq/r^2 \; = 4\pi kq = q/\varepsilon_0$
- (NOTE: **no** r dependence) $k=1/4\pi\varepsilon_0$

- $\nabla \bullet E(r \neq 0) = 0$ – this is true of ANY inverse square field (Gravity also)
- $\nabla \bullet E(r=0) = \delta(r)$ function (∞ at r=0, 0 otherwise)

What if we are not at the center of the sphere?

We break the sphere into two imaginary regions – one sphere inside the other but not centered.
Imagine there is only one charge in the smaller sphere and none between



The total flux when summing over both spheres is ZERO
Since $\nabla \bullet E(r \neq 0) = 0$ in between both spheres (no charges)

$\int \nabla \bullet \mathbf{E}\, dV = 0 = \int E \bullet dA$ (over both spheres)
But $= \int E \bullet dA$ (total) $= 0 = \int E \bullet dA$ (outer sphere) $+ = \int E \bullet dA$ (inner sphere)

Thus:
$\int E \bullet dA$ (outer sphere) $= -\int E \bullet dA$ (inner sphere)

But we know $\int E \bullet dA$ (inner) $= Q/\varepsilon_0$

Hence $\int E \bullet dA$ (outer sphere) $= Q/\varepsilon_0$ (not minus due to the way we oriented the normal to the surface)

There was nothing special about the outer sphere, It could have been any shape, Hence $\int E \bullet dA = Q/\varepsilon_0$ where Q is the total charge enclosed.

More generally $\nabla \bullet E = \rho/\varepsilon_0$

Maxwell Eq #1 of 4 ρ = charge density

Electric Flux

$$d\Phi_E = \mathbf{E} \cdot d\mathbf{A}$$

Differential flux

$$\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{A}$$

Integral flux

$$\Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_S}{\epsilon_0}$$

Flux is charge enclosed Q_s / ε_0

This is Gauss' Law

Coulomb's Law from Gauss' Law

- Assume we have a point charge at the center of a sphere and use Gauss' Law
- And spherical symmetry

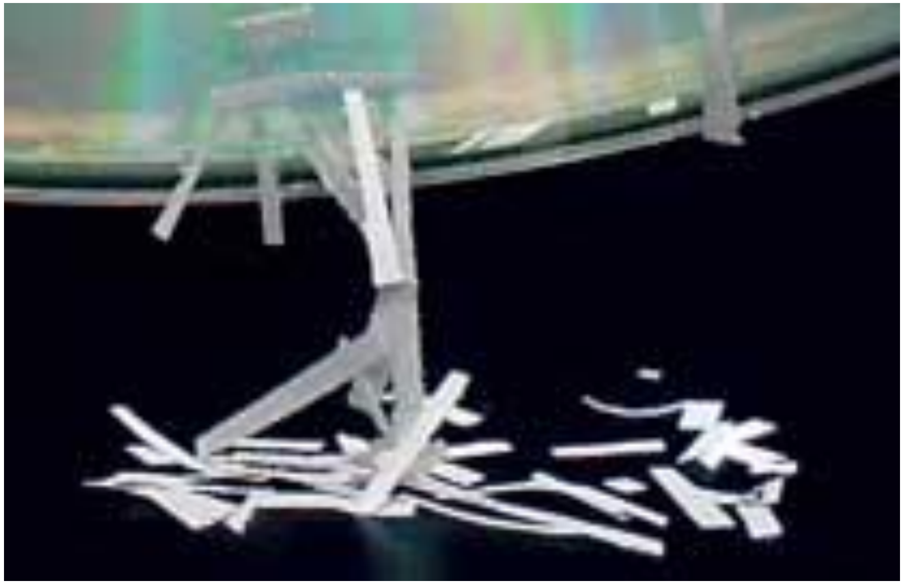
$$\oint_S \mathbf{E} \cdot d\mathbf{A} = Q/\varepsilon_0$$

$$4\pi r^2 \hat{\mathbf{r}} \cdot \mathbf{E}(\mathbf{r}) = Q/\varepsilon_0$$

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0} \frac{\hat{\mathbf{r}}}{r^2}$$

Hence we get Coulomb's Law

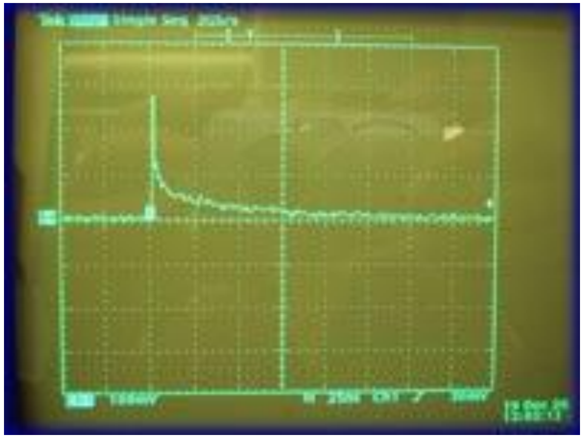
Electrostatics – Here it is dipole moments



Sparks

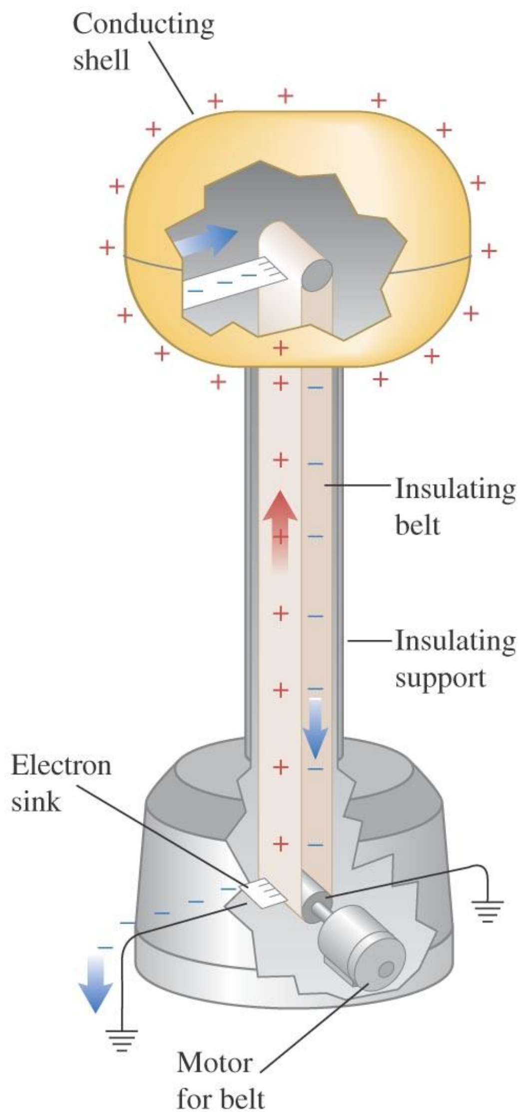
Lightning

Human Lightning

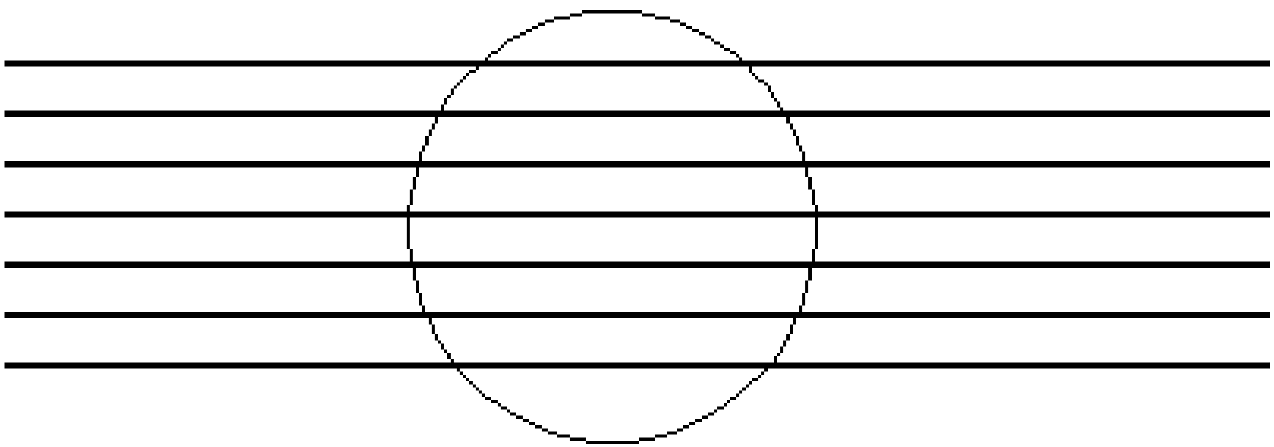


A typical human has a capacitance of about 200-300 pf
Discharge can 10-20 KV+ and amps but microsecond long
Total energy is small so generally not harmful

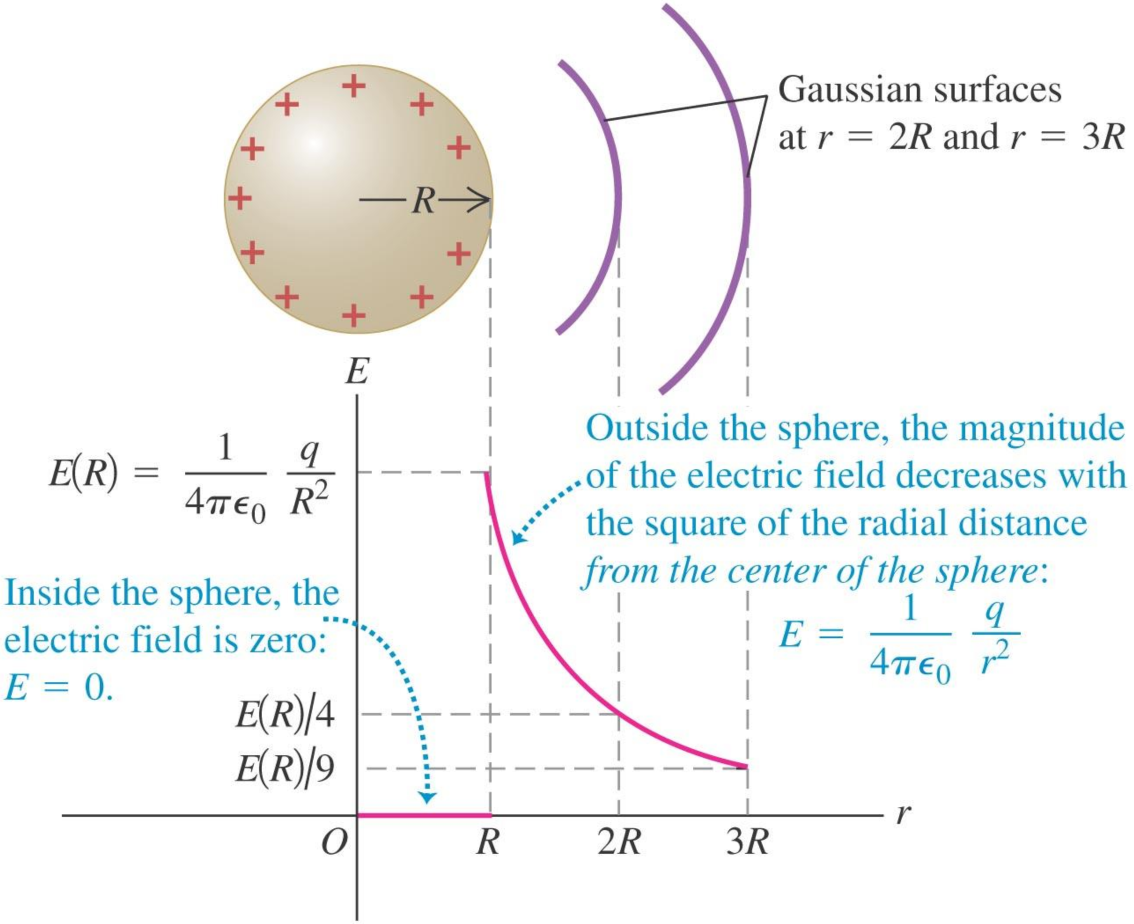
Typical van de Graff generator is Positively charged but NOT always – depends on belt material



Flux =0 through sphere

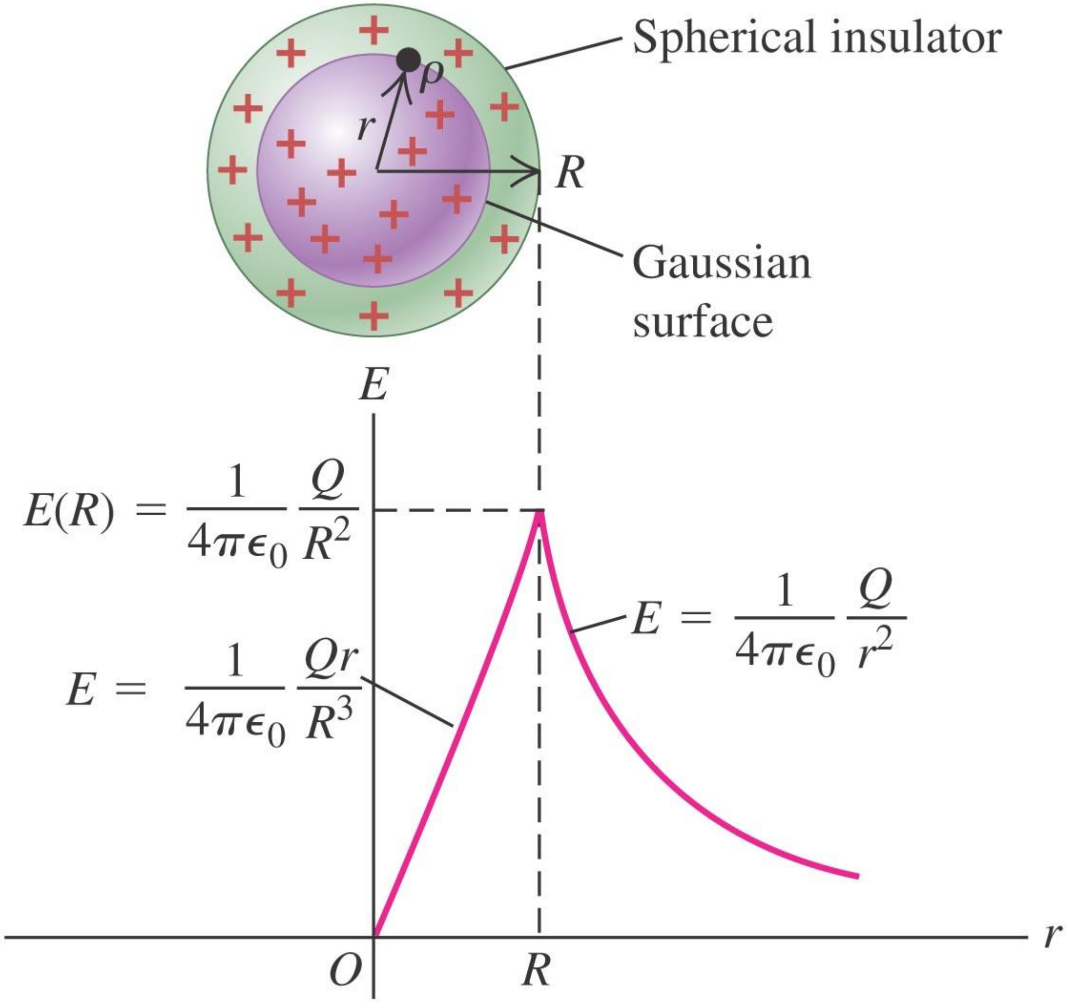


Charged metal sphere – E=0 inside

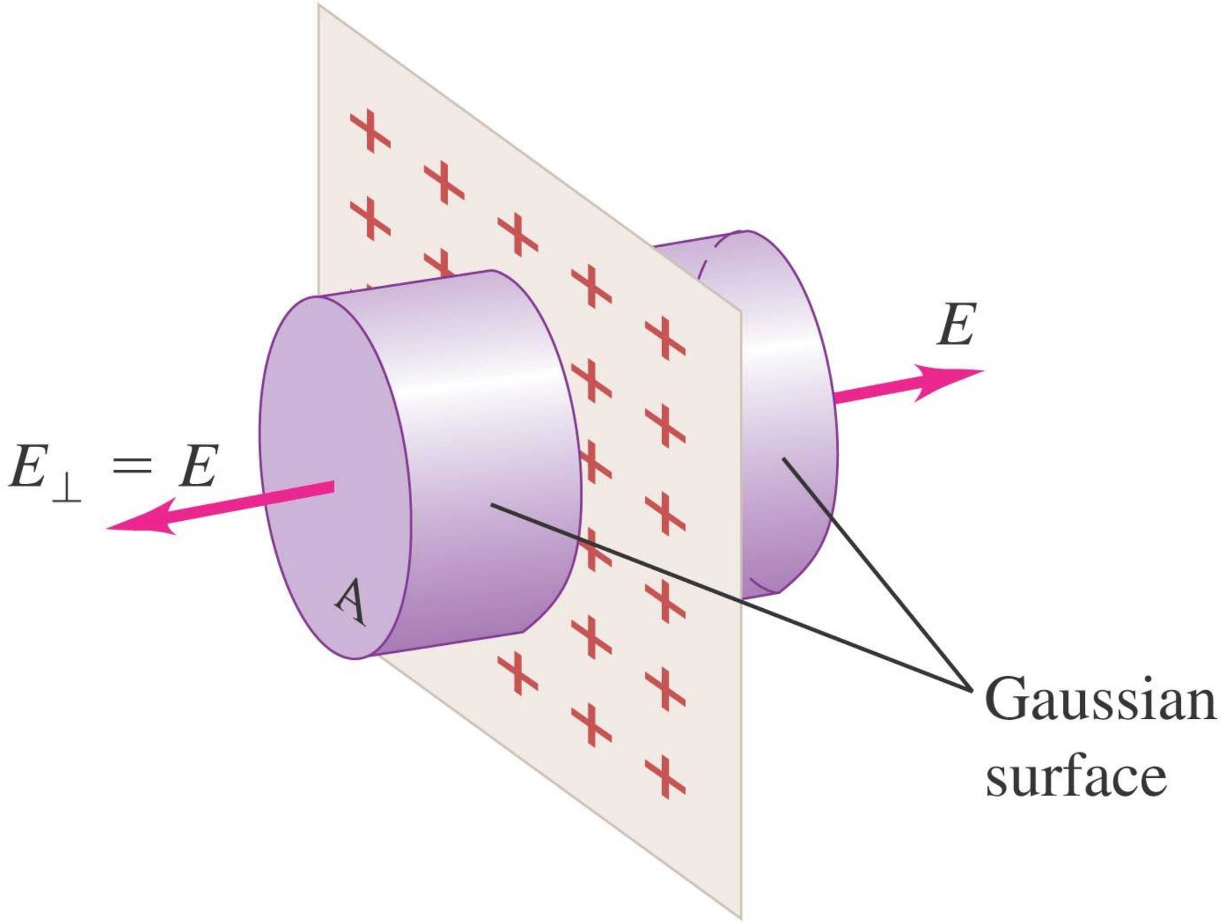


Solving a spherical problem via Gauss' Law

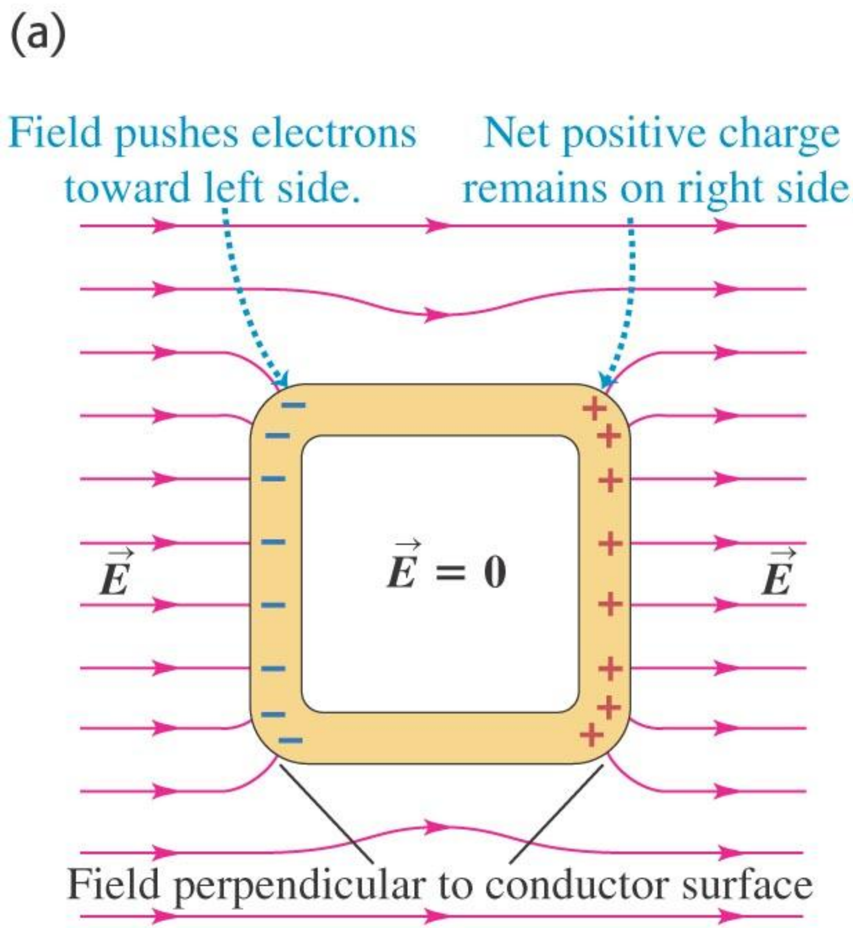
Assume charge Q is spread uniformly over r<R



Gaussian Surface

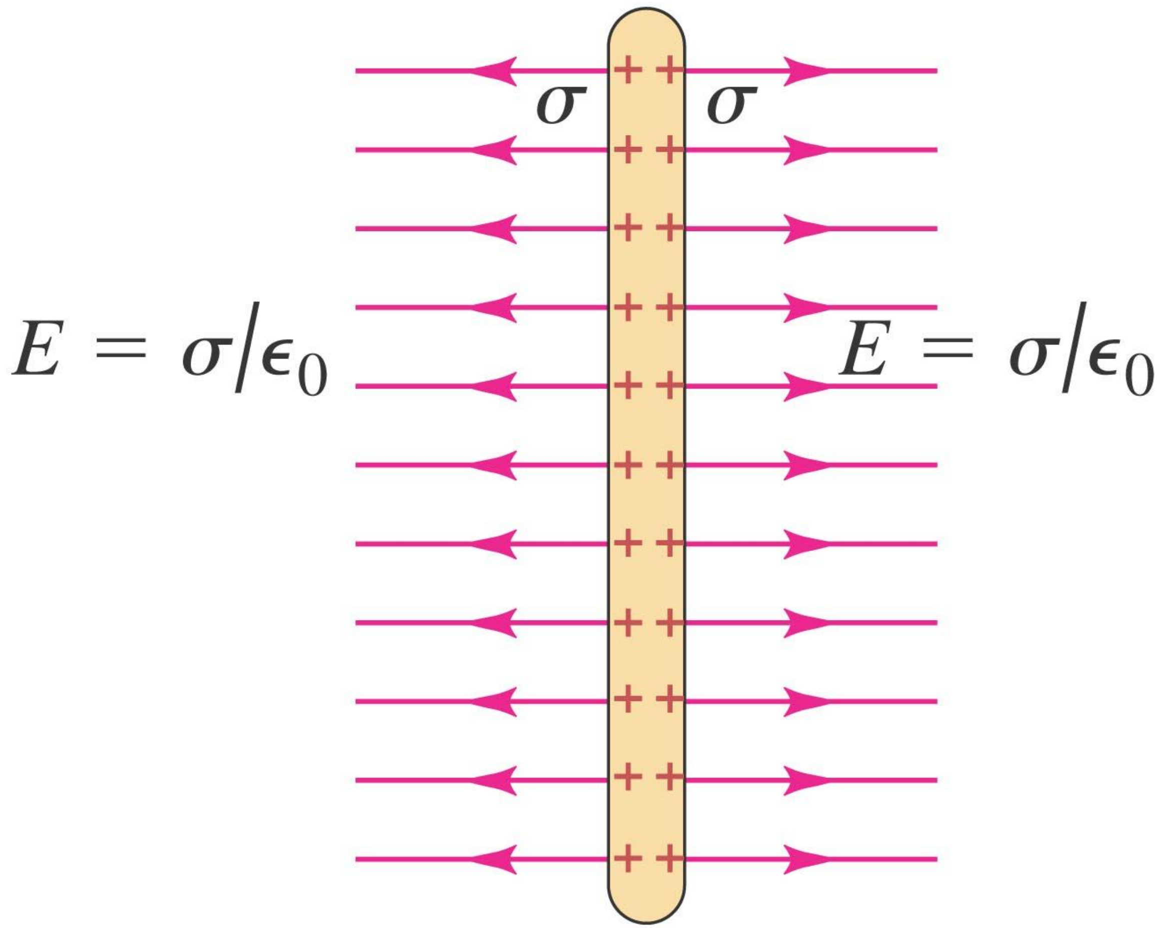


Metal box in external E field
“Faraday Cage”
E=0 inside box

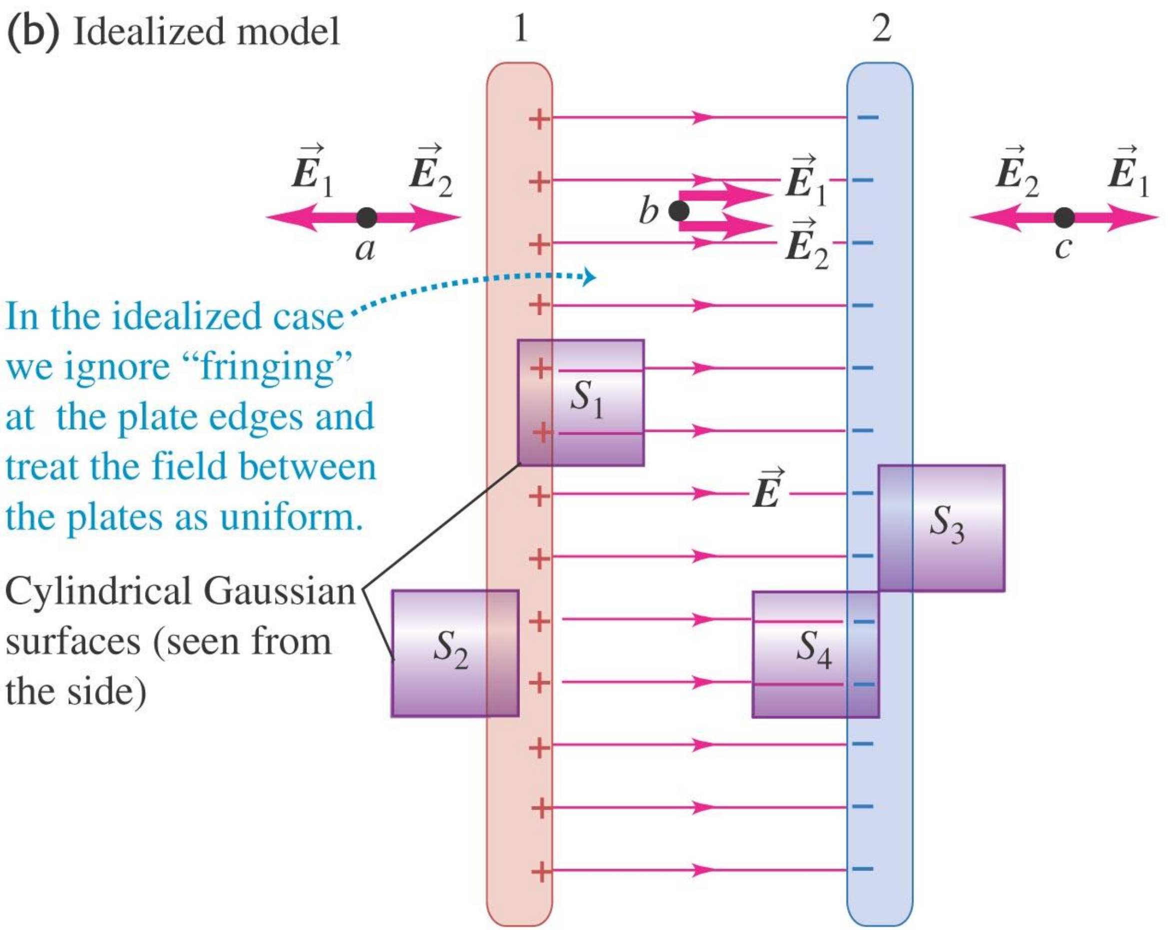


Using a "Gaussian Pillbox" and Gauss' Law to solve for E field from a uniformly charged metal plate with charge per unit area = σ

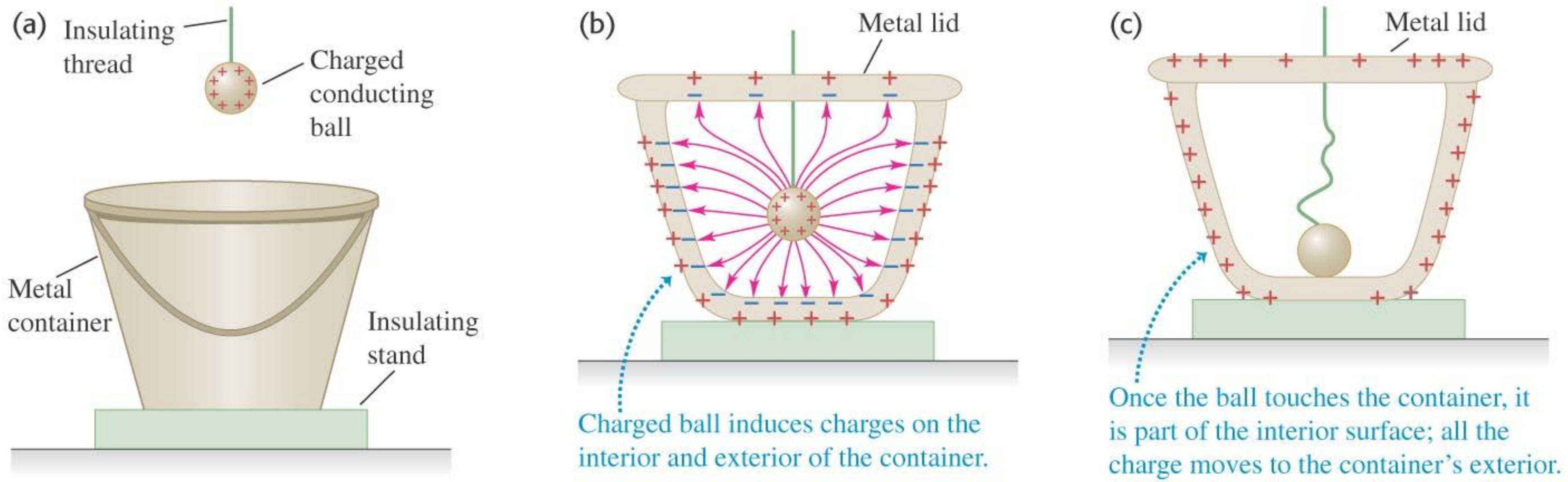
Note E field is the same everywhere except inside metal (=0)



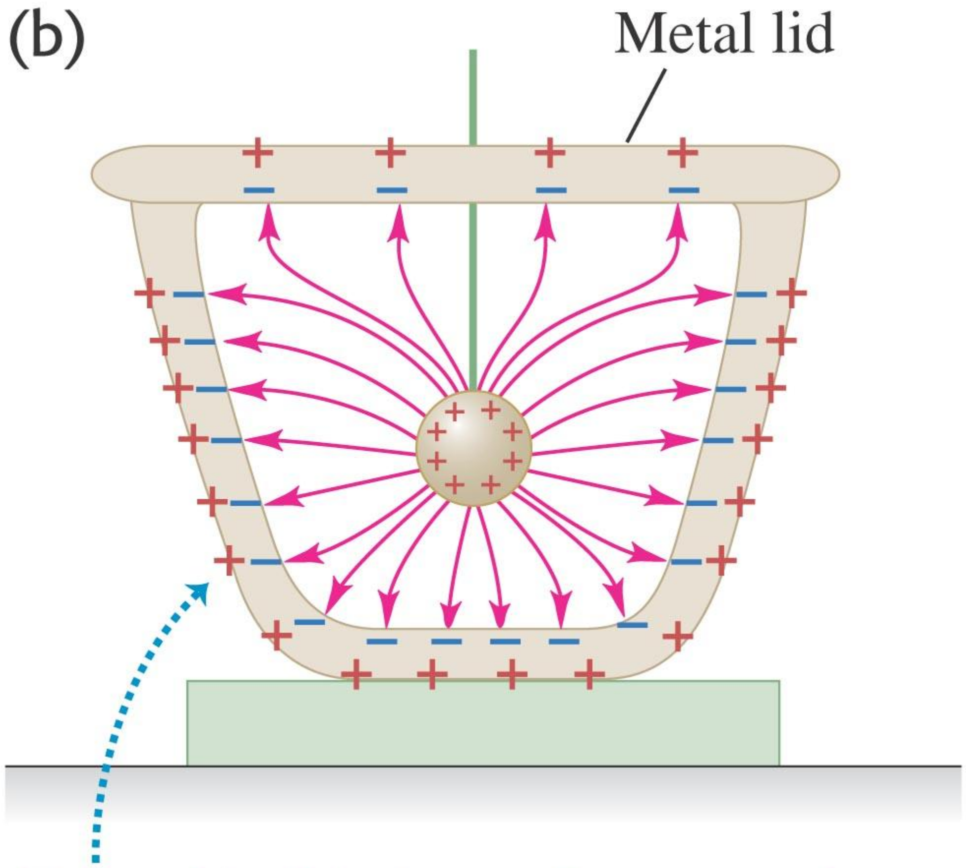
Two metal plates – a Capacitor



Charged ball and metal container

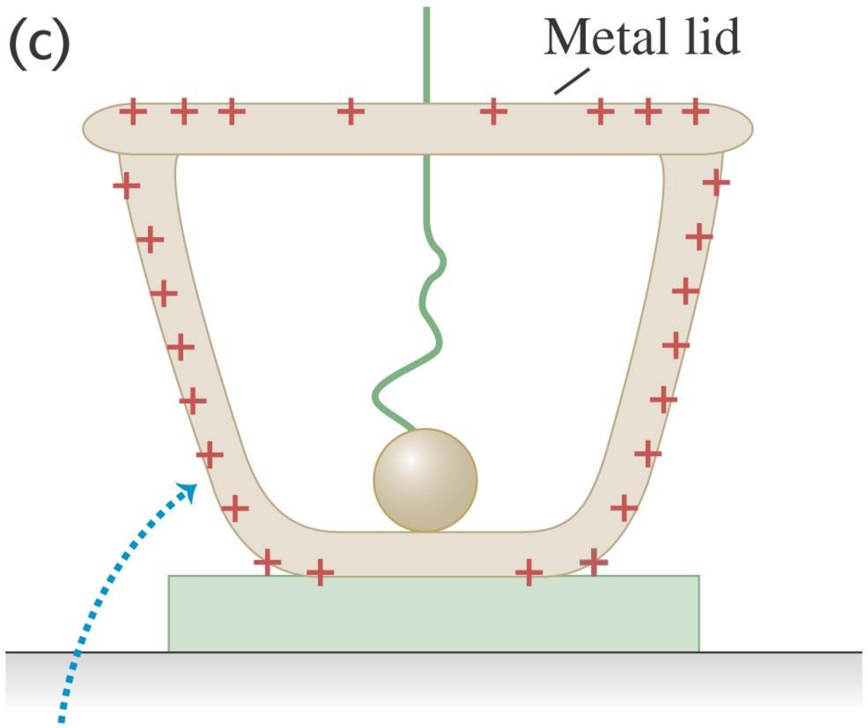


Charged ball inside neutral metal container



Charged ball induces charges on the interior and exterior of the container.

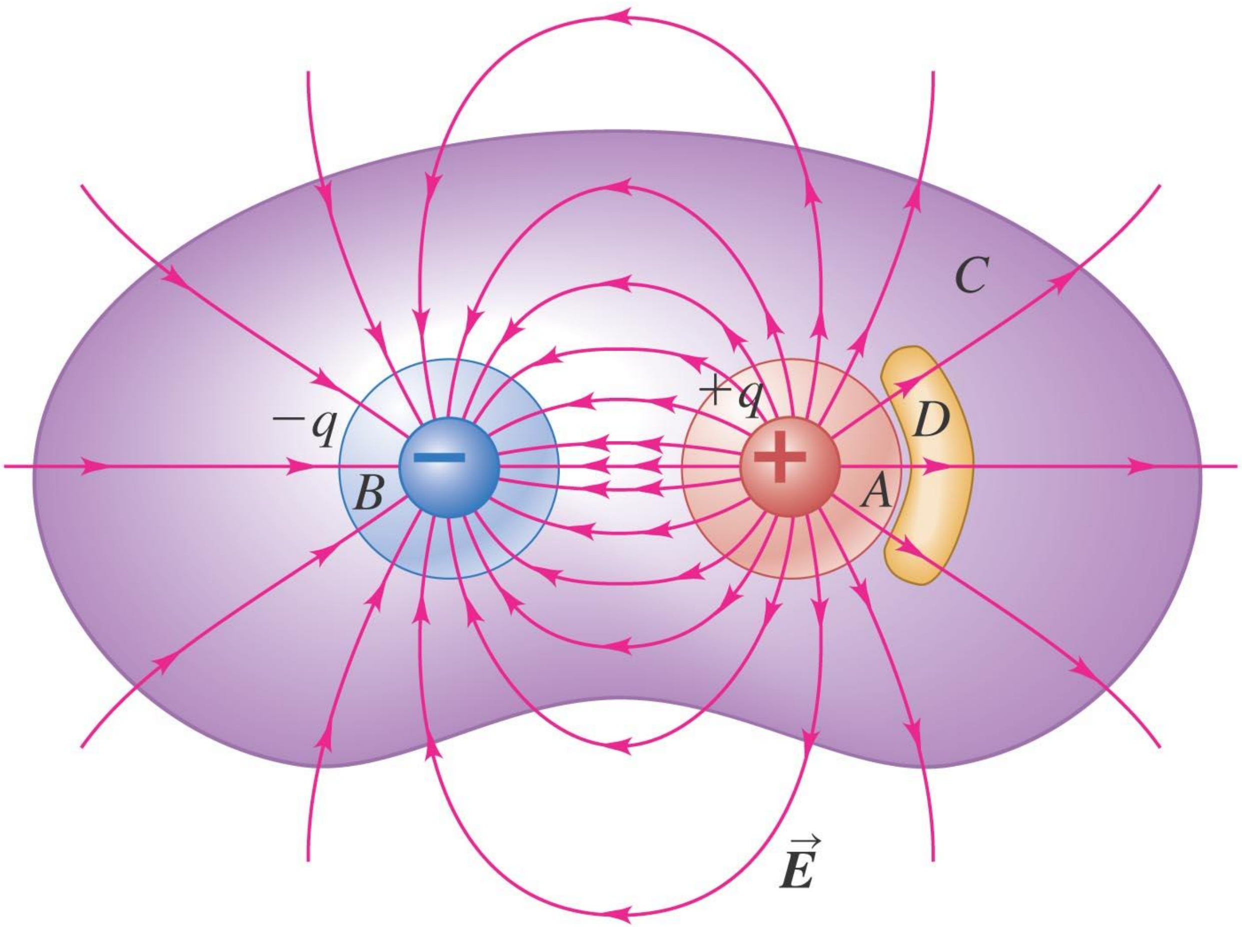
Charge moves to outside of metal container



Once the ball touches the container, it is part of the interior surface; all the charge moves to the container's exterior.

Total flux = total charge enclosed/ε_0

In this case it is ZERO



Lightning

- Approx 16 million lightning storms per year
- Speeds are very high – 60 Km/s (130,000 MPH)!!!
- Temperatures in bolts are very high – can be 30,000 C
- History of kite experiments – wet string = conductive = sparks fly (from key)
- Thomas-François Dalibard and De Lors May 1752
- Benjamin Franklin June 1752 (independent)

Some more on Lightning

- Florida has the most US strikes
- Typ Negative Lightning bolt 30 Kilo amps, 5 Coulombs of charge and 500 Mega Joules of energy
- Large negative bolts can be 120 Kilo amps and 350 Coulombs of charge
- For reference – 1 Ton TNT ~ 4 Giga Joule of energy
- Typ Positive Lightning bolts are 10 times that of Negative Bolts
- Megawatts per meter of bolt are possible
- Typ PEAK power ~ 1 Tera watt (1000 nuclear power plants)
- Lightning heats air to 30,000 C or so and creates supersonic shock wave
- Lightning creates radio waves – these can clear particles from the Van Allen Belts (slots) and create low radiation zones

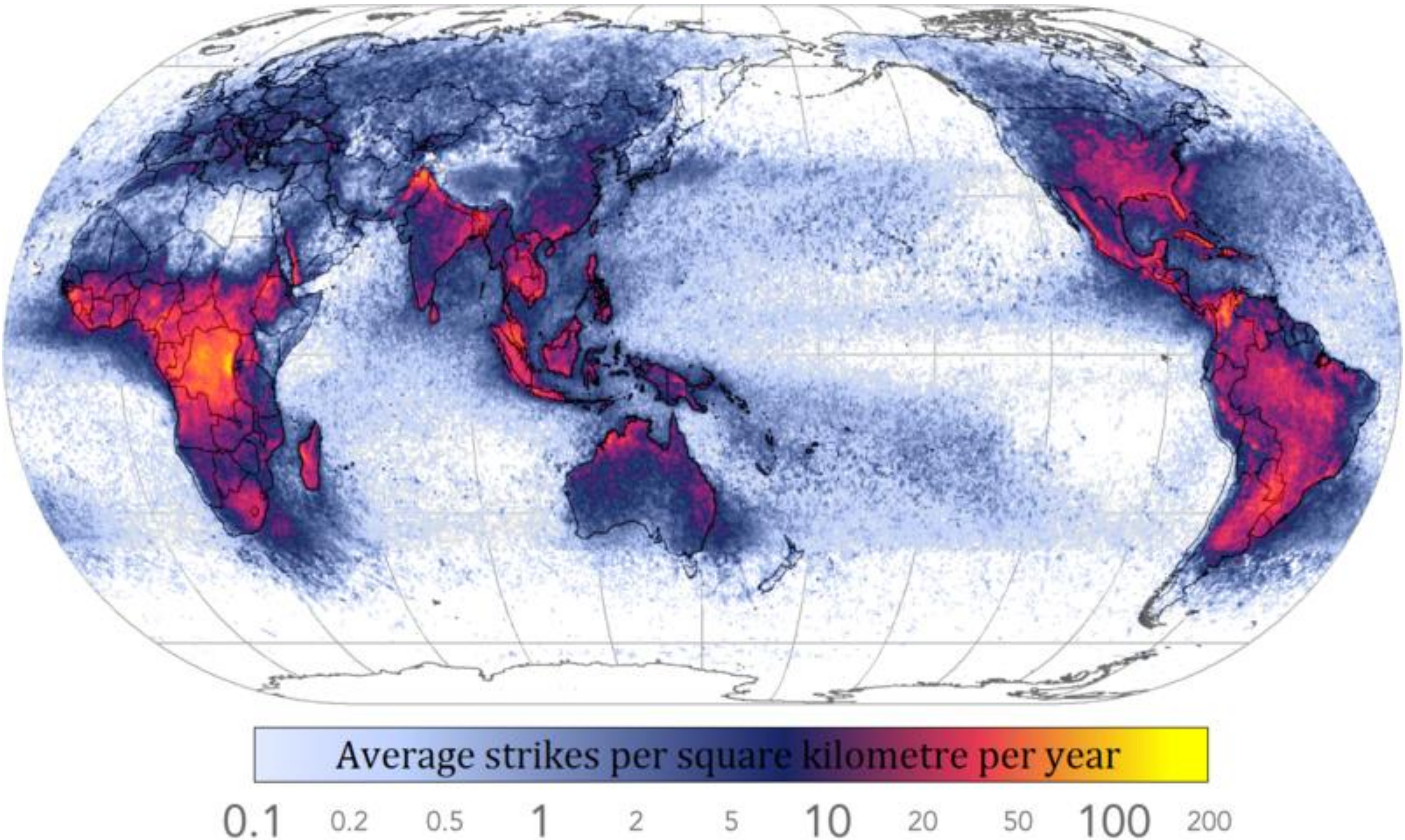
More Lightning Facts

- Norse mythology, Thor is the god of thunder
- Perkūnas - Baltic god of thunder
- Aztec had a god named Tlaloc
- Cyclops gave Zeus the Thunderbolt as a weapon
- Finnish mythology, Ukko (engl. *Old Man*) is the god of thunder
- Brescia, Italy in 1769 – lightning hit the Church of St. Nazaire, igniting 100 tons of gunpoweder that killed 3000 people
- Average lightning hit rate is 44+-5 Hz
- 1.4 billion flashes per year
- Lightning is dangerous - Georg Wilhelm Richmann July 1753 killed as he tried to repeat Ben Franklin's experiment – hit in the head by a blue "ball of lightning"
- Venus, Jupiter and Saturn have lightning

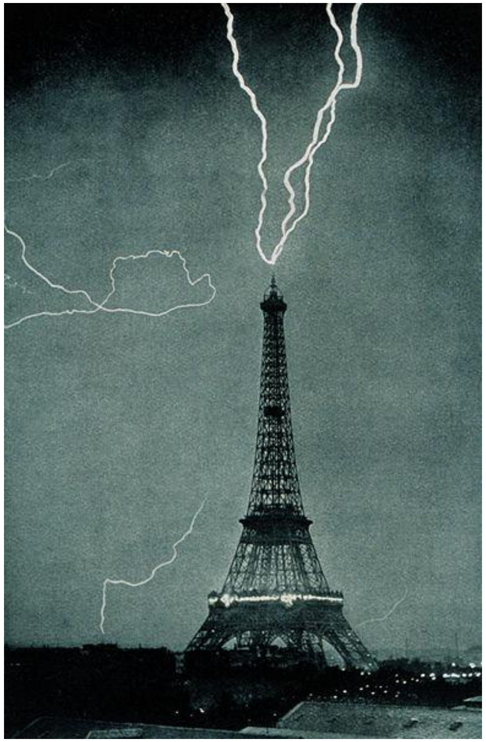
More Lightning Trivia

- Terawatt laser in NM induced minor lightning
- Rockets trailing wires can trigger lightning
- **Elves (Emissions of Light and Very Low Frequency Perturbations from Electromagnetic Pulse Sources**) 250 miles up
- Lighning struck Apollo 12 after take off
- Triggered after above ground nuclear testing
- Triggered by volcanoes
- X-Ray, Gamma Ray 20 Mev and anti matter (positrons) seen from lightning
- Lightning strikes can induce ground magnetic “hot spots”
- Roy Sullivan held a Guinness World Record after surviving 7 different lightning strikes across 35 years.
- October 31 2005, sixty-eight dairy cows, died while taking shelter under a tress on a farm at Fernbrook Dorrigo, New South Wales
- December 8, 1963: Pan Am Flight 214 crashed 81 people were killed.
- November 2, 1994, lightning struck fuel tanks in Dronka, Egypt 469 fatalities

Global Lightning strike distribution



Lightning as Art

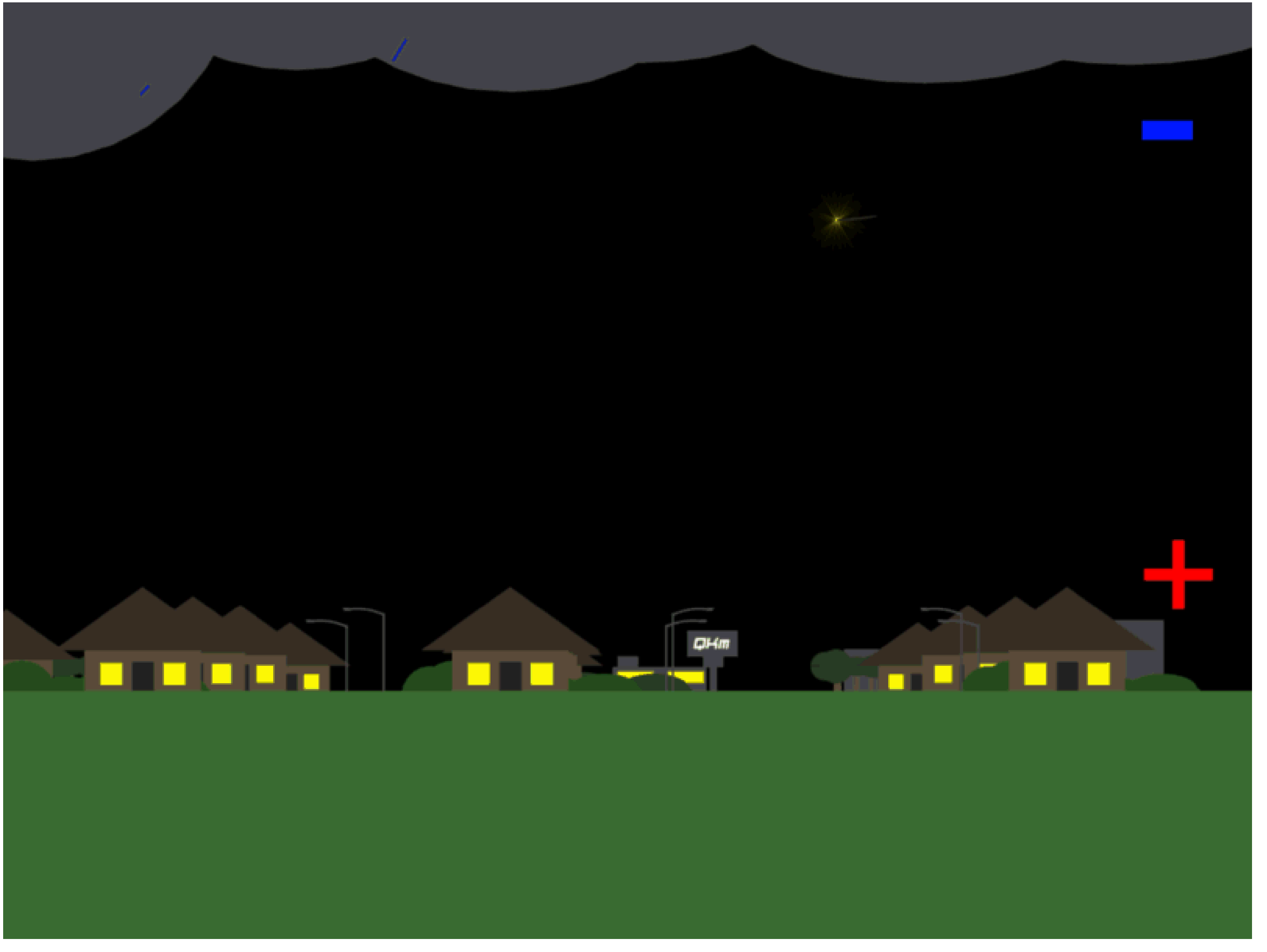


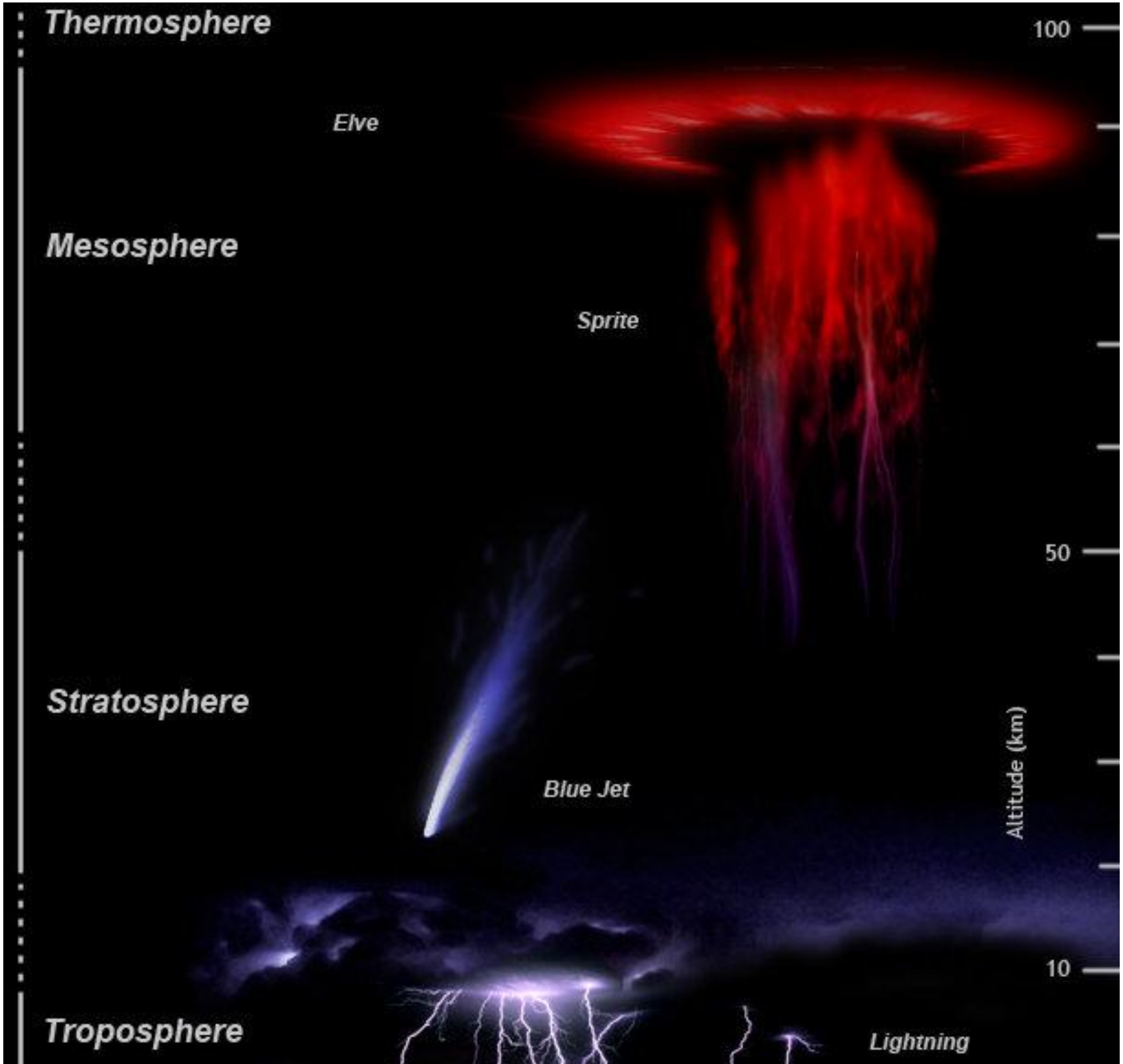
1902 Paris

Rimini Italy

Lightning is complex

Leader first then return strike



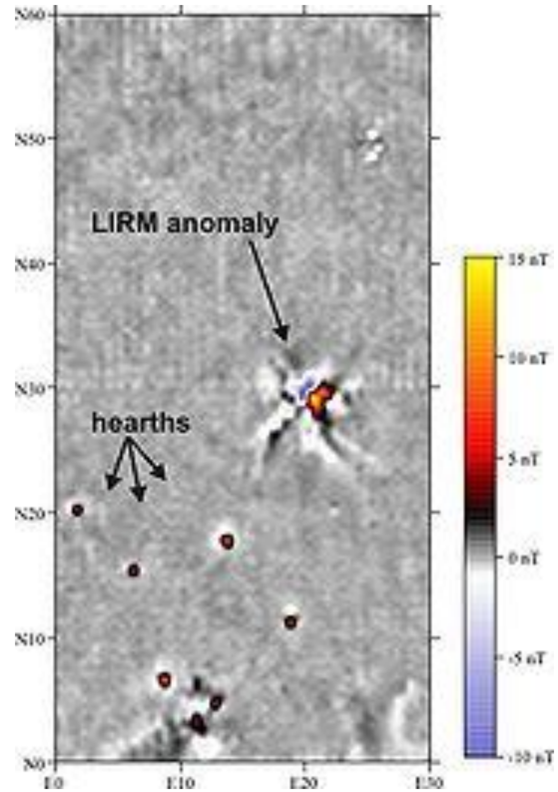

Thermosphere
Elve
Mesosphere
Sprite
Stratosphere
Blue Jet
Troposphere
Lightning
100
50
10
Altitude (km)

Volcano induced lightning

Ball Lightning – Japan

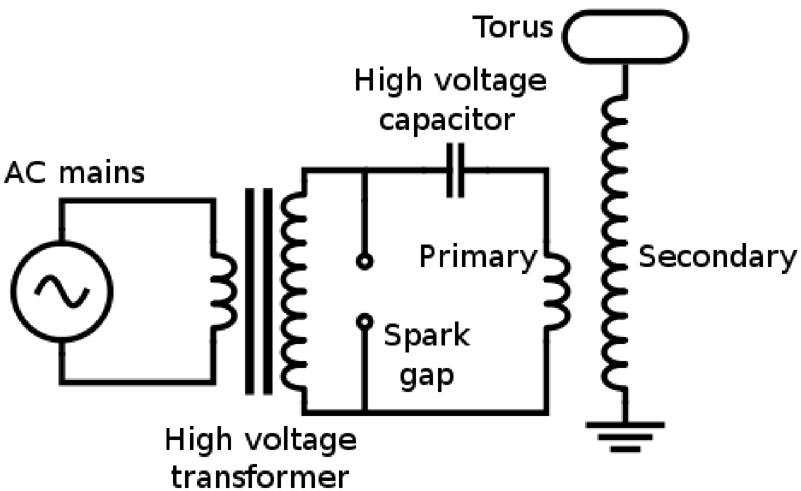
Shuttle hit – STS8 Challenger 1982/3

Cloud to Cloud



Light strike induced magnetism on the ground

Human Made Lightning – Tesla Coils



More Tesla Coil – Note Complex Streamers

Capacitance of simple systems

Type	Capacitance	Comment
Parallel-plate capacitor	$\varepsilon A/d$	A: Area d: Distance
Coaxial cable	$\dfrac{2\pi\varepsilon l}{\ln\left(a_2/a_1\right)}$	a_1: Inner radius a_2: Outer radius *l*: Length
Pair of parallel wires[17]	$\dfrac{2\pi\varepsilon l}{\operatorname{arcosh}\left(\frac{d^2}{2a^2}-1\right)}=\dfrac{\pi\varepsilon l}{\operatorname{arcosh}\left(\frac{d}{2a}\right)}=\dfrac{\pi\varepsilon l}{\ln\left(\frac{d}{2a}+\sqrt{\frac{d^2}{4a^2}-1}\right)}$	a: Wire radius d: Distance, d > 2a *l*: Length of pair
Wire parallel to wall[17]	$\dfrac{4\pi\varepsilon l}{\operatorname{arcosh}\left(\frac{2d^2}{a^2}-1\right)}=\dfrac{2\pi\varepsilon l}{\operatorname{arcosh}\left(\frac{d}{a}\right)}=\dfrac{2\pi\varepsilon l}{\ln\left(\frac{d}{a}+\sqrt{\frac{d^2}{a^2}-1}\right)}$	a: Wire radius d: Distance, d > a *l*: Wire length
Concentric spheres		a_1: Inner radius a_2: Outer radius
Two spheres, equal radius[18][19]	$=2\pi\varepsilon a\left\{\ln 2+\gamma-\frac{1}{2}\ln\left(\frac{d}{a}-2\right)+O\left(\frac{d}{a}-2\right)\right\}$	a: Radius d: Distance, d > 2a D = d/2a γ: Euler's constant
Sphere in front of wall[18]	$4\pi\varepsilon a\sum_{n=1}^{\infty}\dfrac{\sinh\left(\ln\left(D+\sqrt{D^2-1}\right)\right)}{\sinh\left(n\ln\left(D+\sqrt{D^2-1}\right)\right)}$	a: Radius d: Distance, d > a D = d/a
Sphere	$4\pi\varepsilon a$	a: Radius
Circular disc	$8\varepsilon a$	a: Radius
Thin straight wire, finite length[20][21][22]	$\dfrac{2\pi\varepsilon l}{\Lambda}\left\{1+\frac{1}{\Lambda}\left(1-\ln 2\right)+\frac{1}{\Lambda^2}\left[1+\left(1-\ln 2\right)^2-\frac{\pi^2}{12}\right]+O\left(\frac{1}{\Lambda^3}\right)\right\}$	a: Wire radius *l*: Length Λ: ln(*l*/a)