

## Diffracton

### Greens Theorem

Recall divergence theorem

$$\int \nabla \cdot \bar{F} dV = \int \bar{F} \cdot d\bar{\sigma}$$

let  $\bar{F} = U \nabla V - V \nabla U$

$U, V$  scalar, continuous & differentiable

$$\nabla \cdot (U \nabla V) = U \nabla^2 V + \nabla U \cdot \nabla V$$

$$\Rightarrow \nabla \cdot \bar{F} = U \nabla^2 V - V \nabla^2 U$$

$$\Rightarrow \int (U \nabla V - V \nabla U) \cdot d\bar{\sigma} = \int (U \nabla^2 V - V \nabla^2 U) dV$$

Let  $U, V$  satisfy wave equation

$$\nabla^2 U - \frac{1}{v^2} \frac{\partial^2 U}{\partial t^2} = 0$$

$$\nabla^2 V - \frac{1}{v^2} \frac{\partial^2 V}{\partial t^2} = 0$$

if  $U, V$  are harmonic in time  
 $U, V \propto e^{i\omega t}$

$$\begin{aligned}\nabla^2 U &= \frac{1}{v^2} \frac{\partial^2 U}{\partial t^2} = -\frac{\omega^2}{v^2} U \\ \nabla^2 V &= -\frac{\omega^2}{v^2} V\end{aligned}$$

$$\Rightarrow \int (U \nabla V - V \nabla U) \cdot d\bar{\sigma} = 0 \quad \# 1$$

Let  $v$  be a spherical wave solution

$$v = V_0 \frac{e^{ik(r-r_0-wt)}}{r}$$

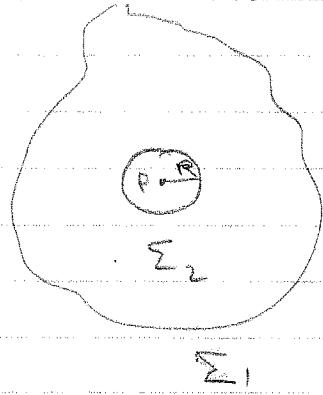
$r=0$  at  $P$

$$\#1 \Rightarrow \int_{\Sigma_1} \left( \frac{e^{ikr}}{r} \nabla v - v \nabla \frac{e^{ikr}}{r} \right) \cdot d\bar{s}$$

$$- \int_{\Sigma_2} = 0$$

$V_0 e^{-iwt}$  taken out

$\Sigma_2$  = sphere centered on  $P$  of radius  $R$



Wave  $v$  converges to  $P$

let  $\Sigma_2$  shrink to a size  $R \rightarrow 0$

$$\lim_{R \rightarrow 0} \int_{\Sigma_2} \frac{e^{ikr}}{r} \nabla v \cdot d\bar{s} = 0$$

$$\int_{\Sigma_2} v \nabla \left( \frac{e^{ikr}}{r} \right) \cdot d\bar{s} = \int_{\Sigma_2} v \frac{\partial}{\partial r} \left( \frac{e^{ikr}}{r} \right) / R^2 dr$$

$$\lim_{R \rightarrow 0} \left[ - \int_{\Sigma_2} v \frac{e^{ikr}}{R^2} dr + \int_{\Sigma_2} v k \frac{e^{ikr}}{R^2} dr \right]$$

$$= - \int_{\Sigma_1} v dr = -4\pi U_p \quad U_p = v(r=0)$$

$$\Rightarrow U_p = -\frac{1}{4\pi} \int_{\Sigma_1} \left( \frac{e^{ikr}}{r} \nabla v - v \nabla \frac{e^{ikr}}{r} \right) \cdot d\bar{s}$$

Kirchhoff theorem

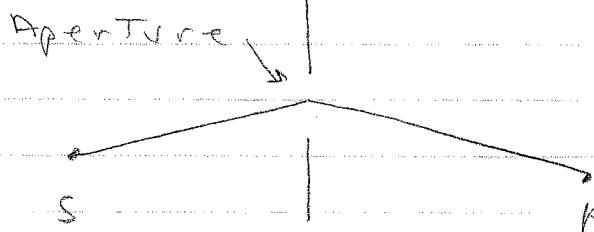
(relates  $U(p)$  to  $U$  on surface  $\Sigma$ )

Since  $E$ ,  $B$  are vectors whereas  $U$ ,  $V$  are scalars this is only an approximation

### Scalar Diffraction

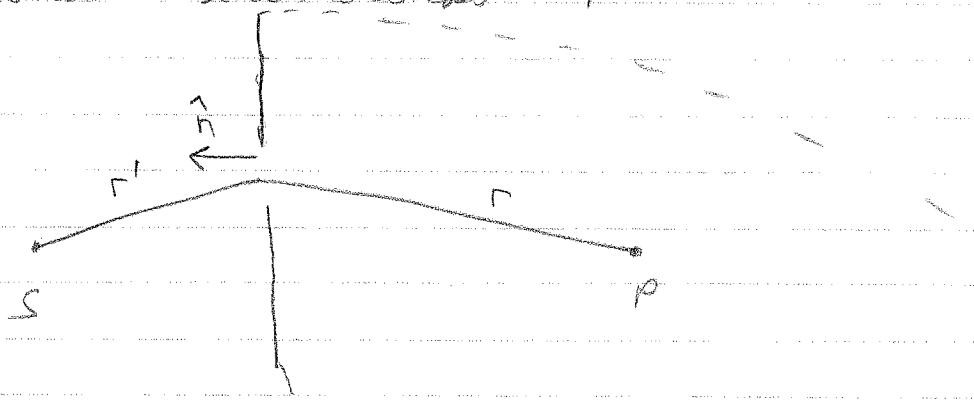
$U =$  "Optical Disturbance"

## Fresnel - Kirchhoff zone



Source Receiver

Now construct a surface  $\Sigma_2$   
that includes the aperture  
and surrounds  $P$



Make two (extreme?) simplifying assumptions ( $P$  or sometimes)

1)  $U \propto D^2$  contribute negligibly  
except at aperture

2)  $U \propto D^2$  are the same at the aperture in absence of partition

If S is a point source  
the U at aperture is:

$$U(r') = U_0 \frac{e^{ikr' - wt}}{r'}$$

$$\Rightarrow U_{Ap} = \frac{U_0 e^{-iwt}}{4\pi} \sum_{\Sigma_{Ap}} \left[ \frac{e^{ikr}}{r} \nabla \left( \frac{e^{ikr'}}{r'} \right) - \frac{e^{ikr'}}{r'} \nabla \left( \frac{e^{ikr}}{r} \right) \right] d\sigma$$

where  $\Sigma_{Ap}$  = Aperture surface

$$\nabla \left( \frac{e^{ikr}}{r} \right) \cdot d\sigma = [e^{ikr} \nabla \left( \frac{1}{r} \right) + \frac{1}{r} \nabla e^{ikr}] \cdot d\sigma$$

$$= \left[ \frac{-e^{ikr}}{r^2} \hat{r} + \frac{ikr}{r} e^{ikr} \right] \cdot d\sigma$$

$$(\nabla \frac{1}{r}) = -\frac{\hat{r}}{r^2}, \quad \nabla r = -\hat{r}, \quad \hat{r} = \hat{n} d\sigma$$

$$= \hat{n} \cdot \hat{r} e^{ikr} \left( \frac{ik}{r} - \frac{1}{r^2} \right)$$

$$\nabla \left( \frac{e^{ikr'}}{r'} \right) \sim \hat{n} \cdot \hat{r}' e^{ikr'} \left( \frac{ik}{r'} - \frac{1}{r'^2} \right)$$

$r, r' \gg$  aperture size

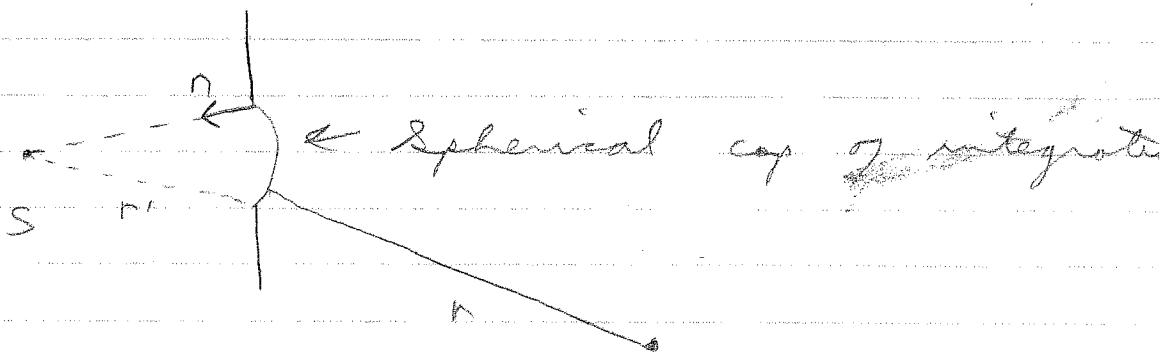
Now assume  $r, r' \gg d$

$$\therefore \frac{k}{r} \approx \frac{1}{rn} \gg \frac{1}{r^2}$$

$$\therefore U_{Ap} \approx -\frac{ikU_0 e^{-wt}}{4\pi} \int \frac{e^{ik(r+n)}}{rn} \cdot \frac{1}{(r-r')} \cdot \hat{n} d\sigma$$

$(\hat{n} - \hat{n}') \cdot \hat{n} =$  Obliquity factor  
Fraunhofer diffraction

Ex i Assume a circular aperture with S symmetrically located



$\hat{n} \cdot \hat{r}' = -1$  over aperture

$$\Rightarrow U_{AP} = -\frac{i k}{4\pi} \int \frac{U_A e^{ik(r-r')}}{r'} (\hat{n} \cdot \hat{r} + 1) d\sigma$$

$$\text{where } U_A = \frac{U_0 e^{ikr}}{r'}$$

$\equiv$  Incident wave at aperture

There is essentially Huygen's principle

Note in forward direction  $\hat{n} \cdot \hat{r} + 1 = 1$

in backward "  $\hat{n} \cdot \hat{r} + 1 = -1$

so no backward wave

also  $-ik \Rightarrow$  diffracted wave  
is phase shifted by  $90^\circ$

Babinet's Principle - Complementarity

Consider two apertures  $A_1, A_2$

Fresnel - Interference is linear

$$\Rightarrow \psi(A) = \psi_1 + \psi_2$$

$$U_p(\text{due to } A) = U_1(p) + U_2(p)$$

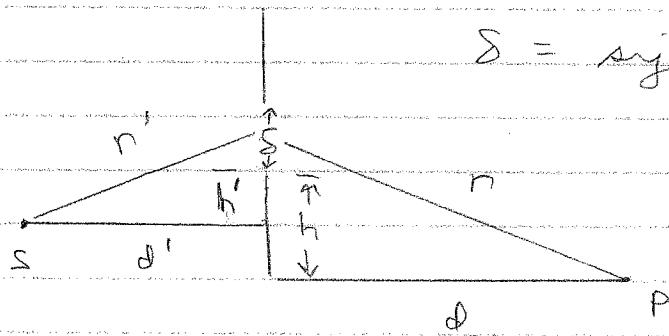
Now if  $A$  is a solid opaque screen then  $U_2 = 0$

$$\Rightarrow U(p) = -U_1(p)$$

π phase shift

## Fraunhofer & Fresnel limits

$\delta = \text{size of aperture}$



How does  $r + r'$  vary over the aperture

$$\Delta = (r + r')_{\text{Top}} - (r + r')_{\text{Bottom}}$$

$$= \sqrt{d'^2 + (h + \delta)^2} + \sqrt{d^2 + (h + \delta)^2} - (\sqrt{d'^2 + h'^2} - \sqrt{d^2 + h^2})$$

$$= \left( \frac{h}{d'} + \frac{h}{d} \right) \delta + \frac{1}{2} \left( \frac{1}{d'} + \frac{1}{d} \right) \delta^2 + \dots$$

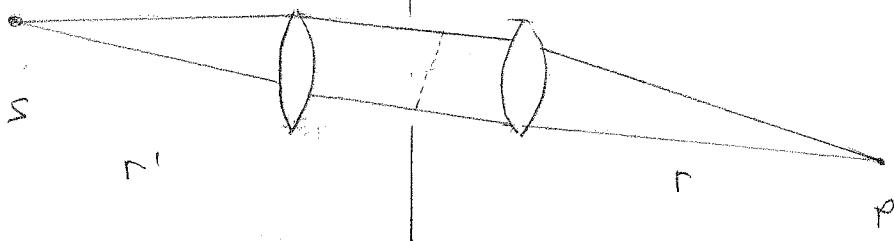
Plane wave      Spherical  
Fraunhofer      Fresnel

Fraunhofer = plane wave at aperture

$$\frac{1}{2} \left( \frac{1}{d'} + \frac{1}{d} \right) \delta^2 \ll \lambda$$

Otherwise Fresnel

## Fraunhofer Diffraction



How to make Fraunhofer

Consequences of Fraunhofer assumption

I  $\hat{n} \cdot \vec{r} - \hat{n} \cdot \vec{r}' = 0$  Obliquity factor  
does not vary significantly  
over aperture  
 $\Rightarrow$  Constant in integral

II  $1/r'$  is nearly constant  
over aperture. (Source is fixed)  
 $\Rightarrow$  Constant in integral

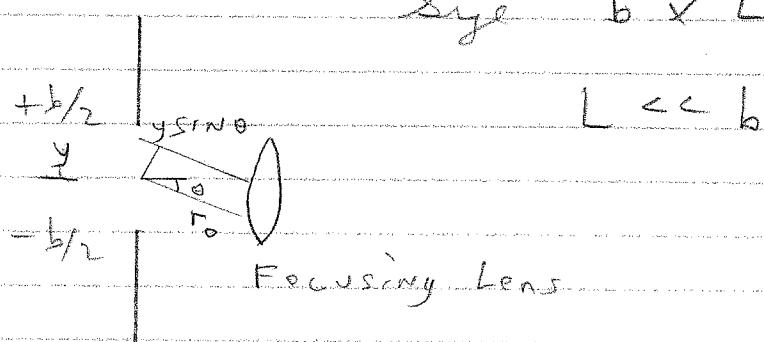
III  $e^{ikr'}$  is varying - if the  
plane wave incident is  $||$   
to the aperture then  $e^{ikr'}$   
is also constant

IV  $1/r$  is nearly constant  
 $e^{ikr}$  varies

$$\Rightarrow U_p (\text{Fraunhofer}) = C \int_{A_p} e^{ik(r+r')} dA$$

# Single Slit (Gaussian)

Size  $b \times L$



$$r = r_0 + y \sin \theta$$

Assume monochromatic plane wave is 11 to aperture so  $r'$  does not vary

$$U = C e^{ikr_0} \int_{-b/2}^{b/2} e^{iky \sin \theta} L dy$$

$$dA$$

$$= 2 C e^{ikr_0} L \frac{\sin(kb \sin \theta)}{k \sin \theta} = C' \frac{\sin \beta}{\beta}$$

$$\beta = \frac{1}{2} kb \sin \theta \quad C' = e^{ikr_0} c b L$$

$$I \propto |U|^2 \propto I_0 \left( \frac{\sin \beta}{\beta} \right)^2$$

Rectangular aperture



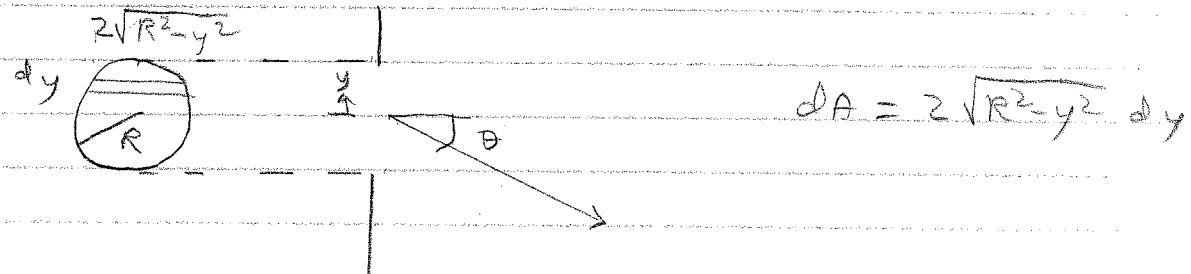
$$I = I_0 \left( \frac{\sin \alpha}{\alpha} \right)^2 \cdot \left( \frac{\sin \beta}{\beta} \right)^2$$

$$\alpha = \frac{1}{\lambda} k a \sin \theta$$

$$\beta = \frac{1}{\lambda} k b \sin \theta$$

## Circular Aperture

Assume incident plane wave II aperture



$$dA = 2\sqrt{R^2 - y^2} dy$$

$$U_p = C e^{ikr_0} \int_{-R}^R e^{iky s \cos \theta} 2\sqrt{R^2 - y^2} dy$$

$$u = y/R, \quad s = kR$$

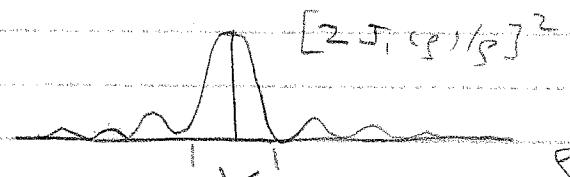
$$\int_{-1}^1 e^{isu} \sqrt{1-u^2} du = \pi J_1(s)/s$$

$J_1$  = Bessel function

$$J_1(s)/s \rightarrow 1/2 \text{ as } s \rightarrow 0$$

$$\Rightarrow I = I_0 \left[ \frac{2J_1(s)}{s} \right]^2$$

$$I_0 = (C\pi R^2)^2$$

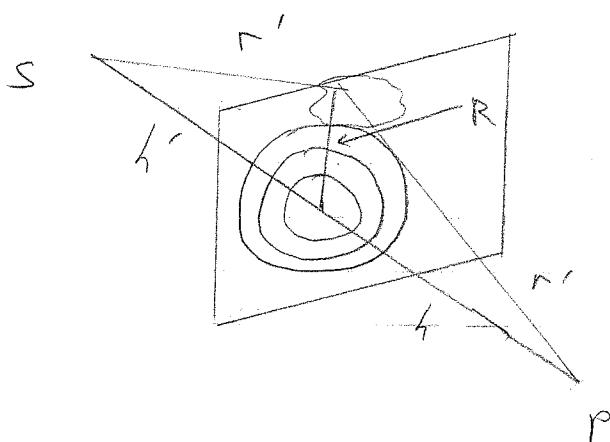


airy Disk

$$\sin \theta = \frac{3.832}{kR} = \frac{1.22\lambda}{D}$$

$$D = 2R$$

## Fresnel diffraction - zones



$$\begin{aligned} r + r' &= (\hbar^2 + R^2)^{1/2} + (\hbar'^2 + R'^2)^{1/2} \\ &= \hbar + \hbar' + \frac{R^2}{2} \left( \frac{1}{\hbar} + \frac{1}{\hbar'} \right) + \dots \end{aligned}$$

Let's divide into concentric zones where  $R = \text{constant}$   
such that  $r + r'$  differs by  $\lambda/2$

$$R_1 = \sqrt{\lambda L}$$

$$R_2 = \sqrt{2\lambda L}$$

$$R_n = \sqrt{n\lambda L}$$

$$L = \left( \frac{1}{\hbar} + \frac{1}{\hbar'} \right)^{-1}$$

These are Fresnel zones

$R_i$ ,  $R_{i+1}$  are inner & outer  
radii of a zone

area of zone

$$A_i = \pi R_{i+1}^2 - \pi R_i^2$$

$$\begin{aligned} &= \pi ((i+1)\lambda L - i\lambda L) \\ &= \pi \lambda L = \pi R_i^2 \end{aligned}$$

All zones have same area

$$V_p = V_1 + V_2 + \dots$$

$$= |V_1| - |V_2| + |V_3| - |V_4| \dots$$

since zones are  $\pi$  but of plus

more

## Zone Plate

Block alternate Fresnel zones

Let block even ones

$$U_p = |U_1| + |U_3| + |U_5| \dots$$

$U_p$ 's are adding magnitudes

$$\rightarrow |U_p| > |U_1|, |U_2| \dots$$

P is bright!

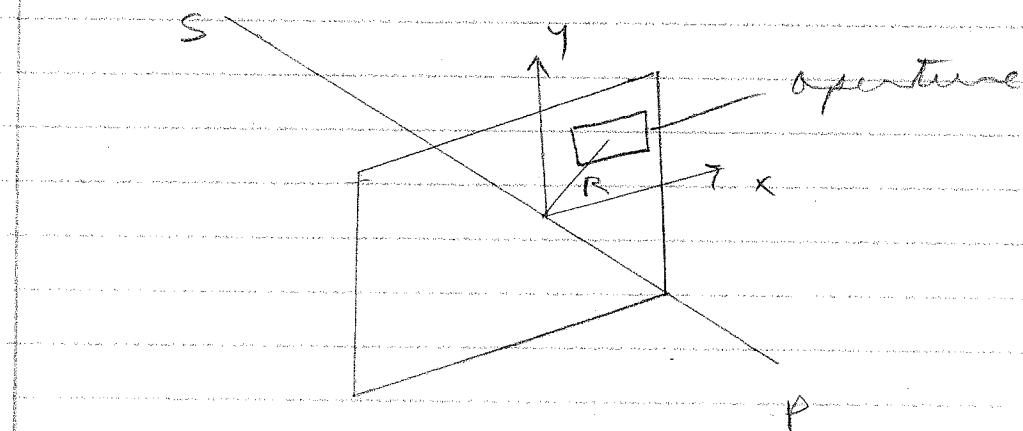
Actually zone plate is a lens

$$\text{Focal length } f = L = R^2/\lambda$$

Examples:

Overhead proj.  
Pocket magnifiers  
"RV" window stickers

## Fresnel Rectangular Aperture



$$R^2 = x^2 + y^2$$

$$c + r' = h + h' + \frac{1}{2}k(x^2 + y^2)$$

$$\sqrt{\frac{R^2}{R^2}}$$

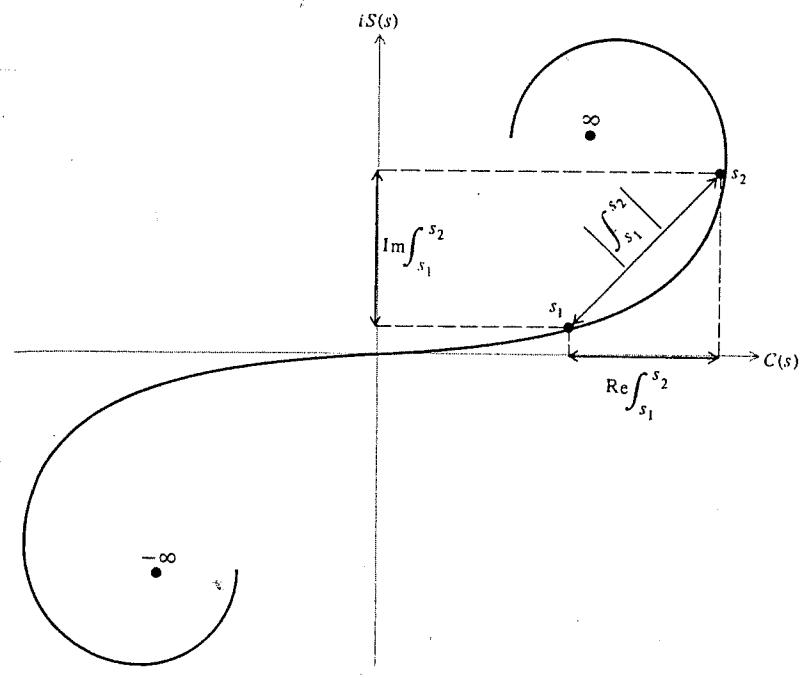
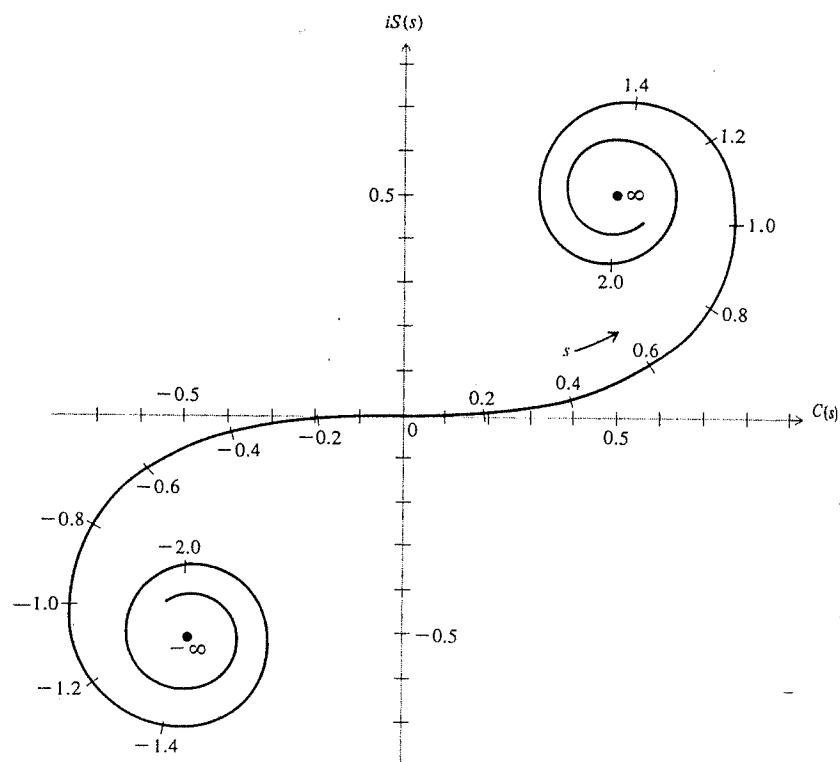
Assume

1)  $\hat{r}, \hat{r}' - \hat{h}, \hat{h}'$  waves slowly over aperture — Obliquity factor

2)  $V_{r,r'}$  waves slowly

$$\Rightarrow U_p = c \int \int e^{ik(x^2 + y^2)/2c} dx dy$$

$$= c \int e^{ikx^2/2c} \int e^{iky^2/2c} dy$$



Define

$$u = x \sqrt{\frac{k}{\pi L}}$$

$$= x \sqrt{\frac{2}{\lambda L}}$$

$$k = \frac{2\pi}{\lambda}$$

$$U_P = U_1 \int_{-u_1}^{u_2} e^{i\pi u^2/2} du \int_{-v_1}^{v_2} e^{i\pi v^2/2} dv$$

$$U_1 = C\pi L / k$$

$$\int_0^s e^{i\pi w^2/2} dw = C(s) + i S(s)$$

Real part  $C(s) = \int_0^s \cos(\pi w^2/2) dw$

Im part  $S(s) = \int_0^s \sin(\pi w^2/2) dw$

These are Fresnel Integrals

$C(s)$  &  $S(s)$  as called  
Cormu Sinal